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The Gradient Projection Method with Exact Line Search*

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Abstract. The gradient projection algorithm for function minimization is often implemented using an approximate local minimization along the projected negative gradient. On the other hand, for some difficult combinational optimization problems, where a starting guess may be far from a solution, it may be advantageous to perform a nonlocal (exact) line search. In this paper we show how to evaluate the piece-wise smooth projection associated with a constraint set described by bounds on the variables and a single linear equation. When the NP hard graph partitioning problem is formulated as a continuous quadratic programming problem, the constraints have this structure.

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1. Introduction

We consider the problem

$$\min_{\mathbf{x}\in\mathcal{K}} f(\mathbf{x}), \tag{1}$$

where $\mathcal{K}$ is a closed, convex subset of $\mathbb{R}^n$, and $f\colon \mathcal{K}\to\mathbb{R}$. The gradient projection method is

$$\mathbf{x}_{k+1} = P(\mathbf{x}_k - s_k \nabla f(\mathbf{x}_k)^{\top}), \tag{2}$$

where $\nabla f(\mathbf{x}_k)$ is the gradient of f at $\mathbf{x}_k$ (the gradient is a row vector), s_k is the stepsize at iteration k, and P denotes projection into $\mathcal{K}$. That is, given $\mathbf{y}\in\mathbb{R}^n$,

$$P(\mathbf{y}) = \arg\min_{\mathbf{x}\in\mathcal{K}} \|\mathbf{y}-\mathbf{x}\|, \tag{3}$$

where $\|\cdot\|$ is the Euclidean norm. Choices for the stepsize include constant stepsize [11, 14], Goldstein's or Armijo's rule [1, 5, 7–10], and exact minimization

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along the projected negative gradient [6, 15]. In the case of exact minimization, the stepsize is given by

$$s_k = \arg\min_{s\geqslant 0} f(P(\mathbf{x}_k - s\nabla f(\mathbf{x}_k)^{\top})). \tag{4}$$

Frequently, exact minimization is not used since it is impractical to evaluate the projection for all choices of s. On the other hand, for difficult optimization problems, where the structure of $\mathcal{K}$ is relatively simple, line search using exact minimization may be useful since it provides a mechanism for making a large step to escape from one valley of the cost function and move to another (possibly distant) valley with a smaller minimum cost.

The NP hard graph partitioning problem is an example of a difficult optimization problem whose constraints are relatively simple. We show in [13] that the problem of partitioning the n nodes of graph into two sets of size m and $n-m$, while minimizing the number of edges that connect the two sets, can be formulated as the following continuous quadratic programming problem:

$$\min\ (\mathbf{1}-\mathbf{x})^{\top}\mathbf{A}\mathbf{x} \quad \text{subject to } \mathbf{0}\leqslant\mathbf{x}\leqslant\mathbf{1}, \quad \mathbf{1}^{\top}\mathbf{x}=m, \tag{5}$$

where $\mathbf{1}$ is the vector containing n ones and $a_{ij}=1$ if and only if either $i=j$ or there is an edge between nodes i and j. This continuous quadratic programming problem has a solution $\mathbf{x}^*$ whose entries are either 0 or 1. The indices of $\mathbf{x}^*$ associated with 1s correspond to a set of m nodes in an optimal partition.

For (5) the constraints consist of a single linear equation and bound constraints on the components of $\mathbf{x}$. In this paper, we show how to evaluated $P(\mathbf{x}+s\mathbf{d})$ as a function of s in this case. For any given s, this projection is the solution of a knapsack problem, which can be solved in time proportional to n[4]. Also, see [16] where an $O(n)$ algorithm is given for more general convex quadratic programs subject to a single linear constraint and bounds on the variables. In this paper, however, we consider a different problem, that of evaluating the projection as a function of s. The projection is a piecewise linear function of s, with a finite set of break points where the derivative with respect to s is discontinuous. We provide an algorithm for computing the break points, and the active indices between break points. The effort involved in computing the new active set after a break point is related to the size of the change in the active set across the break point.

2. Global Convergence

For completeness, we prove that convergent subsequences generated by the gradient projection method with exact line search approach a stationary point. This is proved in [15] for some special cases; also see [6, p. 153] where the analysis is similar to ours, except that our assumptions are local. Here we consider a general closed, convex set $\mathcal{K}$, and a cost function whose gradient is locally Lipschitz. It is well-known [2, p. 201] that the projection operator has the following properties:

$$(\mathbf{y}-P(\mathbf{y}))^\top(\mathbf{x}-P(\mathbf{y}))\leqslant 0 \quad \text{for all } \mathbf{x}\in\mathcal{K}, \tag{6}$$

$$\|P(\mathbf{x})-P(\mathbf{y})\|\leqslant\|\mathbf{x}-\mathbf{y}\| \quad \text{for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^n. \tag{7}$$

The first inequality (6) is the first-order optimality condition satisfied by the solution of (3), while the second inequality (7) is the nonexpansive property of the projection. If f is differentiable along the line segment $[\mathbf{x},\mathbf{y}]$ connecting $\mathbf{x}$ and $\mathbf{y}\in\mathbb{R}^n$, and the following Lipschitz condition holds:

$$\|\nabla f(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))-\nabla f(\mathbf{x})\|\leqslant Lt\|\mathbf{y}-\mathbf{x}\| \quad \text{for all } t\in[0,1],$$

where L is a scalar, then by a second-order Taylor expansion, we have

$$f(\mathbf{y})\leqslant f(\mathbf{x})+\nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x})+\frac{L}{2}\|\mathbf{y}-\mathbf{x}\|^2. \tag{8}$$

THEOREM 1. *Suppose the following conditions hold*:

- *f is differential on $\mathcal{K}$,*
- *the gradient projection iterates* (2)–(4) *are defined,*
- *a subsequence $\{\mathbf{x}_j\colon j\in\mathcal{J}\}$ converges to a limit $\bar{\mathbf{x}}$,*
- *f is Lipschitz continuous in a neighborhood of $\bar{\mathbf{x}}$.*

Then we have

$$\nabla f(\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})\geqslant 0 \quad \textit{for all } \mathbf{x}\in\mathcal{K}.$$

In other words, $\bar{\mathbf{x}}$ is a stationary point for (1).

Proof. If $\nabla f(\bar{\mathbf{x}})=0$, then we are done, so assume that $\nabla f(\bar{\mathbf{x}})\neq 0$. Let $\mathcal{L}$ be a neighborhood of $\bar{\mathbf{x}}$ where f is Lipschitz continuous. Choose L large enough that

$$\|\nabla f(\mathbf{x})-\nabla f(\mathbf{y})\|\leqslant L\|\mathbf{x}-\mathbf{y}\|$$

for all $\mathbf{x}$ and $\mathbf{y}\in\mathcal{L}$. If $\mathcal{B}_\sigma$ is the ball with center $\bar{\mathbf{x}}$ and radius σ, then choose σ small enough that $\mathcal{B}_\sigma\subset\mathcal{L}$ and $\|\nabla f(\mathbf{x})\|\leqslant 2\|\nabla f(\bar{\mathbf{x}})\|$ for all $\mathbf{x}\in\mathcal{B}_\sigma$. Choose L larger, if necessary, so that

$$2\|\nabla f(\bar{\mathbf{x}})\|/L\leqslant\sigma/2. \tag{9}$$

Choose $j\in\mathcal{J}$ large enough that $\mathbf{x}_j\in\mathcal{B}_{\sigma/2}$, and let $\mathbf{y}_j$ denote the projection $P(\mathbf{x}_j-\mathbf{g}_j/L)$, where $\mathbf{g}_j=\nabla f(\mathbf{x}_j)^\top$ and $\bar{\mathbf{g}}=\nabla f(\bar{\mathbf{x}})^\top$. Since $\mathbf{x}_j\in\mathcal{K}$, we have $P(\mathbf{x}_j)=\mathbf{x}_j$, and by (7) and (9),

$$\|\mathbf{y}_j-\mathbf{x}_j\|=\|P(\mathbf{x}_j-\mathbf{g}_j/L)-P(\mathbf{x}_j)\|\leqslant\|\mathbf{g}_j\|/L\leqslant 2\|\bar{\mathbf{g}}\|/L\leqslant\sigma/2.$$

Since $\mathbf{x}_j\in\mathcal{B}_{\sigma/2}$, it follows that $\mathbf{y}_j\in\mathcal{B}_\sigma$. By (8) and the relation $f(\mathbf{x}_{k+1})\leqslant f(\mathbf{y}_k)$ for each k, we have

$$\begin{aligned} f(\mathbf{x}_{j+1})-f(\mathbf{x}_j) &\leqslant f(\mathbf{y}_j)-f(\mathbf{x}_j)\\ &\leqslant \nabla f(\mathbf{x}_j)(\mathbf{y}_j-\mathbf{x}_j)+\frac{L}{2}\|\mathbf{y}_j-\mathbf{x}_j\|^2. \end{aligned} \tag{10}$$

The optimality condition (6) for the projection, with $\mathbf{y}=\mathbf{x}_j-\mathbf{g}_j/L$, can be expressed:

$$(\mathbf{x}_j-\mathbf{g}_j/L-\mathbf{y}_j)^\top(\mathbf{x}-\mathbf{y}_j)\leqslant 0 \quad \text{for all } \mathbf{x}\in\mathcal{K}.$$

Taking $\mathbf{x}=\mathbf{x}_j$ yields:

$$\nabla f(\mathbf{x}_j)(\mathbf{y}_j-\mathbf{x}_j)\leqslant -L\|\mathbf{y}_j-\mathbf{x}_j\|^2. \tag{11}$$

Combining this with (10) gives

$$\|\mathbf{y}_j-\mathbf{x}_j\|^2\leqslant \frac{2}{L}(f(\mathbf{x}_j)-f(\mathbf{x}_{j+1})). \tag{12}$$

Since $f(\mathbf{x}_k)$ is a monotone decreasing function of k and since $f(\mathbf{x}_j)$, $j\in\mathcal{J}$, approaches $f(\bar{\mathbf{x}})$, it follows, that the entire sequence $f(\mathbf{x}_k)$ converges monotonically to $f(\bar{\mathbf{x}})$. As a result, (12) implies that

$$\lim_{j\in\mathcal{J}}\mathbf{y}_j=\bar{\mathbf{x}}. \tag{13}$$

By the nonexpansive property (7) of the projection operator, we have

$$\begin{aligned}\|\mathbf{y}_j-P(\bar{\mathbf{x}}-\bar{\mathbf{g}}/L)\| &= \|P(\mathbf{x}_j-\mathbf{g}_j/L)-P(\bar{\mathbf{x}}-\bar{\mathbf{g}}/L)\|\\ &\leqslant \|\mathbf{x}_j-\mathbf{g}_j/L-(\bar{\mathbf{x}}-\bar{\mathbf{g}}/L)\|\\ &\leqslant \|\mathbf{x}_j-\bar{\mathbf{x}}\|+\|\mathbf{g}_j-\bar{\mathbf{g}}\|/L. \end{aligned}\tag{14}$$

By the Lipschitz continuity of ∇f, the right side of (14) approaches 0. Combining this with the convergence (13) of $\mathbf{y}_j$ to $\bar{\mathbf{x}}$, the left side yields:

$$\lim_{j\in\mathcal{J}}\|\mathbf{y}_j-P(\bar{\mathbf{x}}-\bar{\mathbf{g}}/L)\|=\|\bar{\mathbf{x}}-P(\bar{\mathbf{x}}-\bar{\mathbf{g}}/L)\|=0.$$

It follows that

$$\bar{\mathbf{x}}=P(\bar{\mathbf{x}}-\bar{\mathbf{g}}/L).$$

Finally, (6) with $\mathbf{y}=\bar{\mathbf{x}}-\bar{\mathbf{g}}/L$ completes the proof. □

3. Structure of the Projection

Given $\mathbf{y}_0$ and $\mathbf{d}\in\mathbb{R}^n$, let $\mathbf{y}(\alpha)$ be the affine function of $\alpha\in\mathbb{R}$ defined by

$$\mathbf{y}(\alpha)=\mathbf{y}_0+\alpha\mathbf{d}.$$

Let $\mathbf{x}(\alpha)$ be the solution of the problem

$$\min_{\mathbf{x}\in\mathbb{R}^n}\frac{1}{2}\|\mathbf{x}-\mathbf{y}(\alpha)\|^2 \quad \text{subject to } \mathbf{0}\leqslant\mathbf{x}\leqslant\mathbf{1},\ \mathbf{a}^\top\mathbf{x}=b, \tag{15}$$

where $\mathbf{a}\in\mathbb{R}^n$ and $b\in\mathbb{R}$. We assume that y_0 satisfies the constraints of (15). That is, $\mathbf{0}\leqslant\boldsymbol{y}_0\leqslant\mathbf{1}$ and $\mathbf{a}^\mathsf{T}\mathbf{y}_0=b$. Without loss of generality, we can assume that $\mathbf{a}\geqslant\mathbf{0}$; that is, for each i such that $a_i<0$, we make the change of variables given by $\bar{x}_i=1-x_i$, while $\bar{x}_i=x_i$ otherwise. After modifying y in the same way and writing the resulting problem in the form (15), the new $\mathbf{a}$ is nonnegative. If $a_i = 0$ for some i, then x_i does not appear in the linear constraint of (15). Since the ith term in the cost function is independent of the other terms, we see that when $a_i = 0$,

$$x_i(\alpha) = \begin{cases} 0 & \text{if } y_i(\alpha) < 0, \\ 1 & \text{if } y_i(\alpha) > 1, \\ y_i(\alpha) & \text{otherwise.} \end{cases}$$

Since these components of $\mathbf{x}(\alpha)$ corresponding to indices i where a_i vanishes can be expressed independently of the other components, we simplify the discussion further by assuming that $\mathbf{a} > \mathbf{0}$.

We now formulate the first-order optimality system associated with (15). Define the index sets

$$\begin{aligned} \mathcal{U}(\alpha) &= \{i \in [1, n]: \ x_i(\alpha) = 1\}, \\ \mathcal{L}(\alpha) &= \{i \in [1, n]: \ x_i(\alpha) = 0\}, \\ \mathcal{F}(\alpha) &= \{i \in [1, n]: \ 0 < x_i(\alpha) < 1\}. \end{aligned}$$

Since the first-order optimally system is necessary and sufficient for optimality in this convex setting, it follows that $\mathbf{x} = \mathbf{x}(\alpha)$ achieves the minimum in (15) if and only if $\mathbf{0} \leqslant \mathbf{x}(\alpha) \leqslant \mathbf{1}$ and there exist $\lambda(\alpha) \in \mathbb{R}$ such that

$$x_i(\alpha) - y_i(\alpha) + a_i \lambda(\alpha) \begin{cases} \leqslant 0 & \text{if } i \in \mathcal{U}(\alpha), \\ \geqslant 0 & \text{if } i \in \mathcal{L}(\alpha), \\ = 0 & \text{if } i \in \mathcal{F}(\alpha). \end{cases} \tag{16}$$

It follows from (16) that $x_i(\alpha) = y_i(\alpha) - a_i \lambda(\alpha)$ for each $i \in \mathcal{F}(\alpha)$. If $\mathcal{F}(\alpha) \neq \emptyset$, then we can substitute for $x_i(\alpha)$, $i \in \mathcal{F}(\alpha)$, in the constraint $\mathbf{a}^\top \mathbf{x} = b$ and solve for $\lambda(\alpha)$:

$$\lambda(\alpha) = \frac{\sum_{i \in \mathcal{F}(\alpha)} a_i y_i(\alpha) + \sum_{i \in \mathcal{U}(\alpha)} a_i - b}{\sum_{i \in \mathcal{F}(\alpha)} a_i^2}. \tag{17}$$

If $\mathcal{F}(\alpha) = \emptyset$, then in general, there is a closed interval of multipliers satisfying the optimality conditions. More precisely, when $\mathcal{F}(\alpha) = \emptyset$, the equalities in (16) are vacuous, and since $a_i > 0$ for each i, the inequalities imply that

$$\lambda(\alpha) \leqslant (y_i(\alpha) - x_i(\alpha))/a_i \quad \text{for each } i \in \mathcal{U}(\alpha), \tag{18}$$

$$\lambda(\alpha) \geqslant (y_i(\alpha) - x_i(\alpha))/a_i \quad \text{for each } i \in \mathcal{L}(\alpha). \tag{19}$$

The smallest upper bound combined with the largest lower bound yields an interval $[\lambda_l(\alpha), \lambda_u(\alpha)]$ of possible multipliers (if one of the bounds is infinite, the interval becomes semi-infinite). We let $\Lambda(\alpha)$ denote the set of multipliers satisfying (18) and (19), while $\lambda(\alpha) \in \Lambda(\alpha)$ denotes a multiplier chosen from the set.

LEMMA 1. *If* $\mathcal{U}(\alpha_1)=\mathcal{U}(\alpha_2)$ *and* $\mathcal{L}(\alpha_1)=\mathcal{L}(\alpha_2)$, *then* $\mathcal{U}(\alpha)=\mathcal{U}(\alpha_1)$ *and* $\mathcal{L}(\alpha)=\mathcal{L}(\alpha_1)$ *for all* $\alpha\in[\alpha_1,\alpha_2]$.

Proof. Let $\bar{\mathbf{x}}(\cdot)$: $\mathbb{R}\to\mathbb{R}^n$ and $\bar{\lambda}(\cdot)$: $\mathbb{R}\to\mathbb{R}$ be affine functions satisfying the conditions

$$\bar{\mathbf{x}}(\alpha)=\mathbf{x}(\alpha) \quad \text{and} \quad \bar{\lambda}(\alpha)\in\Lambda(\alpha), \quad \alpha=\alpha_1,\alpha_2.$$

Since $\mathcal{U}(\alpha_1)=\mathcal{U}(\alpha_2)$ and $\mathcal{L}(\alpha_1)=\mathcal{L}(\alpha_2)$, it follows that for all $\alpha\in[\alpha_1,\alpha_2]$, $\bar{\mathbf{x}}_i(\alpha)=1$ if $i\in\mathcal{U}(\alpha_1)$, $\bar{\mathbf{x}}_i(\alpha)=0$ if $i\in\mathcal{L}(\alpha_1)$, and

$$0<\bar{\mathbf{x}}_i(\alpha)<1 \quad \text{if } i\in\mathcal{F}(\alpha_1).$$

Since $\bar{\mathbf{x}}$, $\bar{\lambda}$, and $\mathbf{y}$ satisfy (16) for $\alpha=\alpha_1$ and $\alpha=\alpha_2$, we have for all $\alpha\in[\alpha_1,\alpha_2]$,

$$\bar{x}_i(\alpha)-y_i(\alpha)+a_i\bar{\lambda}(\alpha)\begin{cases}\leqslant 0 & \text{if } i\in\mathcal{U}(\alpha_1),\\ \geqslant 0 & \text{if } i\in\mathcal{L}(\alpha_1),\\ =0 & \text{if } i\in\mathcal{F}(\alpha_1).\end{cases}$$

Hence, $\bar{\mathbf{x}}(\alpha)$ satisfies the first-order optimality conditions associated with (15) and $\mathbf{x}(\alpha)=\bar{\mathbf{x}}(\alpha)$ for each $\alpha\in[\alpha_1,\alpha_2]$. □

Remark 1. By [12, (4.4)], $\mathbf{x}(\alpha)$ depends Lipschitz continously on α and the following estimate holds:

$$\|\mathbf{x}(\alpha_1)-\mathbf{x}(\alpha_2)\|\leqslant|\alpha_1-\alpha_2|\|\mathbf{d}\|. \tag{20}$$

Suppose that at some $\beta\in\mathbb{R}$, $\mathcal{F}(\beta)$ is empty and $\lambda_l(\beta)<\lambda_u(\beta)$. Since $\mathbf{x}$ and $\mathbf{y}$ depend continuously on α in (18) and (19), $\lambda_l(\alpha)<\lambda_u(\alpha)$ for α near β. Hence, $\mathbf{x}(\beta)$ is the solution of (15) for α near β.

Although $\Lambda(\alpha)$ can be a set when $\mathcal{F}(\alpha)$ is empty , $\Lambda(\alpha)$ must be a singleton when $\mathcal{F}(\alpha)\neq\emptyset$. That is, for any $i\in\mathcal{F}(\alpha)$ (16) implies that

$$\lambda(\alpha)=(y_i(\alpha)-x_i(\alpha))/a_i.$$

We now show that Λ possesses continuity properties, despite the fact that it can switch between a single-valued and a multi-values function. We say that β is a break point if either $\mathcal{L}(\beta-\epsilon)\neq\mathcal{L}(\beta+\epsilon)$ or $\mathcal{U}(\beta-\epsilon)\neq\mathcal{U}(\beta+\epsilon)$ for all ϵ sufficiently close to 0. We attach a + or - superscript to a set to denote its value just to the right or just to the left of its evaluation point. For example, $\mathcal{F}^-(\beta)$ and $\mathcal{F}^+(\beta)$ denote the free set just to the left and just to the right of β, respectively.

LEMMA 2. *The multiplier set has the following properties*:

(a) *If* $\mathcal{F}^{+}(\beta)\neq\emptyset\neq\mathcal{F}^{-}(\beta)$ *and* $\mathcal{F}(\beta)=\emptyset$, *then* $\Lambda(\cdot)$ *is single-valued and continuous at* β.
(b) *If* $\mathcal{F}(\beta)=\mathcal{F}^{-}(\beta)=\emptyset$, *but* $\mathcal{F}^{+}(\beta)\neq\emptyset$ *then* $\Lambda(\cdot)$ *is single-valued and continuous on an interval* $[\beta,\beta+\epsilon]$ *for some* $\epsilon>0$.
(c) *On any interval* $[\alpha_1,\alpha_2]$ *where* Λ *is single-valued, it is Lipschitz continuous.*

Proof. In case (a), if $\Lambda(\beta)$ is multivalued, then as observed above, $\mathbf{x}(\alpha)=\mathbf{x}(\beta)$ for α near β, and $\mathcal{F}(\alpha)$ is empty for α near β. Since this contradicts the assumption that $\mathcal{F}^{+}(\beta)\neq\emptyset\neq\mathcal{F}^{-}(\beta)$, $\Lambda(\beta)$ is single-valued, and $\lambda_l(\beta)=\lambda_u(\beta)$. For α near β, it follows from the continuity of $\mathbf{x}(\cdot)$, that for each $i\in\mathcal{U}(\beta)$, either $i\in\mathcal{U}(\alpha)$ or $i\in\mathcal{F}(\alpha)$. In either case, (16) implies that

$$\lambda(\alpha)\leqslant(y_i(\alpha)-x_i(\alpha))/a_i \quad \text{for each } i\in\mathcal{U}(\beta). \tag{21}$$

Let j be any index in $\mathcal{U}(\beta)$ for which $\lambda_u(\beta)=(y_j(\beta)-x_j(\beta))/a_j$. Utilizing (21) with $i=j$ and (20) gives

$$\lambda(\alpha)\leqslant\lambda_u(\beta)+2\|\mathbf{d}\||\alpha-\beta|/a_j.$$

In the same fashion, but with $\mathcal{U}$ replaced by $\mathcal{L}$, we have

$$\lambda(\alpha)\geqslant\lambda_l(\beta)-2\|\mathbf{d}\||\alpha-\beta|/a_j.$$

As α approaches β, we deduce that $\lambda(\alpha)$ approaches $\lambda_u(\beta)=\lambda_l(\beta)=\lambda(\beta)$. This completes the proof of (a). In case (b), the same analysis can be used to conclude that $\Lambda(\beta)$ is single-valued and $\Lambda(\alpha)$ approaches $\Lambda(\beta)$ as α approaches β from the right.

In case (c), let $(\beta_1,\beta_2)\subset(\alpha_1,\alpha_2)$ be any interval that contains no break points. Suppose that $\alpha\in(\beta_1,\beta_2)$, $\mathcal{F}(\alpha)$ is nonempty, and $j\in\mathcal{F}(\alpha)$. By (16) we have $\lambda(\alpha)=(y_j(\alpha)-x_j(\alpha))/a_j$ for all $\alpha\in(\beta_1,\beta_2)$. Due to the Lipschitz estimate(20), λ is Lipschitz continuous on (α_1,α_2). If $\mathcal{F}(\alpha)$ is empty for all $\alpha\in(\alpha_1,\alpha_2)$, then by the assumption that Λ is single-valued, by the bounds (18) and (19), and by the Lipschitz continuity of $\boldsymbol{x}$ and $\boldsymbol{y}$, we conclude, again, that λ is Lipschitz continuous on (α_1,α_2). Finally, let β be any break point in (α_1,α_2). If $\mathcal{F}(\beta)$ is nonempty, then the relation $\lambda(\alpha)=(y_j(\alpha)-x_j(\alpha))/a_j$ for any $j\in\mathcal{F}(\beta)$ and α near β implies that λ is continuous at β. If $\mathcal{F}(\beta)$ is empty, then by either (a) or (b), we have continuity of λ at β. Lipschitz continuity between break points combined with continuity across break points yields (c). □

As a consequence of Lemma 1, there is a finite set of break points. Let β_1 and β_2 be adjacent break points. If $\mathcal{F}(\cdot)$ is empty on (β_1,β_2), then $\mathbf{x}(\cdot)$ is constant.

If $\mathcal{F}(\cdot)$ is nonempty, then $\lambda(\cdot)$ is affine due to (17), while $x_i(\cdot)$ for $i \in \mathcal{F}(\alpha)$, $\alpha \in (\beta_1, \beta_2)$, is affine due to (16). Moreover, we have

$$\lambda'(\alpha) = \frac{\sum_{i \in \mathcal{F}(\alpha)} a_i d_i}{\sum_{i \in \mathcal{F}(\alpha)} a_i^2} \tag{22}$$

and

$$x_i'(\alpha) = d_i - a_i \lambda'(\alpha) \quad \text{for each } i \in \mathcal{F}(\alpha). \tag{23}$$

Hence, at the break point β_2, either some component of $x_i(\cdot)$ that was previously free reaches a bound, or one of the inequalities in (16) changes from strict inequality to equality. In other words, at β_2 one of the following conditions holds:

$$x_i(\beta_2) = 0 \text{ or } 1 \quad \text{for some } i \in \mathcal{F}^+(\beta_1), \text{ or} \tag{24}$$

$$x_i(\beta_2) = y_i(\beta_2) - a_i \lambda(\beta_2), \; i \in \mathcal{A}^+(\beta_1) \quad \text{with } d_i \neq a_i \lambda'(\alpha). \tag{25}$$

Here $\mathcal{A}(\alpha) = \mathcal{L}(\alpha) \cup \mathcal{U}(\alpha)$ are the active indices at α. Given any interval, where $\mathcal{A}(\cdot)$ is constant, it is easy to determine the break point to the right by checking when either (24) or (25) is first satisfied. We now examine the more difficult problem of determining the active set $\mathcal{A}^+(\beta_2)$ just to the right of the break point β_2.

4. Active Set Transition at a Break Point

In this section, we give an algorithm for evaluating $\mathcal{A}^+(\beta)$ either at a break point β or at the starting point $\beta = 0$. If $\beta = 0$ and $\mathcal{F}(\beta)$ is empty with $\lambda_l(0) < \lambda_u(0)$, then as noted in Section 3, $\mathcal{L}^+(0) = \mathcal{L}(0)$ and $\mathcal{U}^+(0) = \mathcal{U}(0)$. In the case that $\Lambda(\beta)$ is single-valued, the algorithm for evaluating $\mathcal{A}^+(\beta)$ involves a nested sequence of sets $\{\mathcal{U}_m\}$ and $\{\mathcal{L}_m\}$ with the following initializations:

$$\mathcal{U}_0 = \{i \in \mathcal{U}(\beta): \; x_i(\beta) - y_i(\beta) + a_i \lambda(\beta) < 0\}, \tag{26}$$

$$\mathcal{L}_0 = \{i \in \mathcal{L}(\beta): \; x_i(\beta) - y_i(\beta) + a_i \lambda(\beta) > 0\}. \tag{27}$$

For any m, we also define

$$\mathcal{F}_m = (\mathcal{U}_m \cup \mathcal{L}_m)^c \quad \text{and} \quad \bar{d}_m = \frac{\sum_{i \in \mathcal{F}_m} a_i d_i}{\sum_{i \in \mathcal{F}_m} a_i^2}. \tag{28}$$

Here lower case superscript 'c' is used to denote the ordinary complement relative to the entire set of indices. We use an upper case superscript 'C' to denote a restricted complement, relative to the set $\mathcal{U}(\beta)$ or $\mathcal{L}(\beta)$, defined in the following way:

$$\mathcal{U}_m^C = \{i \in \mathcal{U}(\beta): \, i \notin \mathcal{U}_m, d_i \geqslant a_i \bar{d}_m\},$$

$$\mathcal{L}_m^C = \{i \in \mathcal{L}(\beta): \, i \notin \mathcal{L}_m, d_i \leqslant a_i \bar{d}_m\}.$$

The new sets $\mathcal{U}_{m+1}$ and $\mathcal{L}_{m+1}$ are computed as follows:

$$\mathcal{U}_{m+1} = \mathcal{U}_m \cup \mathcal{U}_m^C \quad \text{and} \quad \mathcal{L}_{m+1} = \mathcal{L}_m \quad \text{if } \sigma_m \geqslant \tau_m, \tag{29}$$

$$\mathcal{L}_{m+1} = \mathcal{L}_m \cup \mathcal{L}_m^C \quad \text{and} \quad \mathcal{U}_{m+1} = \mathcal{U}_m \quad \text{otherwise}, \tag{30}$$

where

$$\sigma_m = \sum_{i \in \mathcal{U}_m^C} a_i d_i - a_i^2 \bar{d}_m \quad \text{and} \quad \tau_m = \sum_{i \in \mathcal{L}_m^C} a_i^2 \bar{d}_m - a_i d_i. \tag{31}$$

When the summation limits in (31) are empty, we define the sum to be -1. That is, $\sum_{i \in \emptyset} = -1$. Since $d_i \geqslant a_i \bar{d}_m$ in the definition of σ_m and since $d_i \leqslant a_i \bar{d}_m$ in the definition of τ_m, it follows that both σ_m and τ_m are nonnegative when their limits are nonempty. The construction is terminated at the first value of m, denoted M, for which both $\mathcal{U}_m^C$ and $\mathcal{L}_m^C$ are empty.

THEOREM 2. *If either β is a break point, or $\beta = 0$ and $\Lambda(\beta)$ is a singleton, then*

$$\mathcal{U}^+(\beta) = \mathcal{U}_M \quad \textit{and} \quad \mathcal{L}^+(\beta) = \mathcal{L}_M.$$

Thus to obtain $\mathcal{U}^+(\beta)$ and $\mathcal{L}^+(\beta)$, we let the sets $\mathcal{U}_m$ and $\mathcal{L}_m$ grow until both $\mathcal{U}_m^C$ and $\mathcal{L}_m^C$ are empty. Let M_1 be the parameter defined by:

$$M_1 = \begin{cases} M & \text{if } \mathcal{F}_M \text{ is nonempty}, \\ M-1 & \text{otherwise}. \end{cases}$$

Note that although $\mathcal{U}_M$ and $\mathcal{L}_M$ are defined, $\bar{d}_M$ is undefined when $\mathcal{F}_M$ is empty since it reduces to 0/0. The following lemma, which is the basis for Theorem 2, establishes a relationship between $\bar{d}_m$ and $\bar{d}_k$, $k \in (m, M_1]$.

LEMMA 3. *If $\sigma_m \geqslant \tau_m$, then $\bar{d}_m \geqslant \bar{d}_k$ for all $k \in (m, M_1]$, and if $\sigma_m < \tau_m$, then $\bar{d}_m \leqslant \bar{d}_k$ for all $k \in (m, M_1]$.*

Proof. We begin with the cases corresponding to $k = m+1 \leqslant M_1$. That is, we prove the following:

(P1) If $\sigma_m \geqslant \tau_m$, then $\bar{d}_m \geqslant \bar{d}_{m+1}$.
(P2) If $\sigma_m < \tau_m$, then $\bar{d}_m \leqslant \bar{d}_{m+1}$.

Since $\mathbf{a} > \mathbf{0}$, the relation $d_j \geqslant a_j \bar{d}_m$ for each $j \in \mathcal{U}_m^C$ implies that

$$a_j d_i \geqslant a_j^2 \bar{d}_m = a_j^2 \frac{\sum_{i \in \mathcal{F}_m} a_i d_i}{\sum_{i \in \mathcal{F}_m} a_i^2}.$$

Summing over $j \in \mathcal{U}_m^C$ gives

$$\sum_{j \in \mathcal{U}_m^C} a_j d_j \geqslant \left(\sum_{j \in \mathcal{U}_m^C} a_j^2 \right) \frac{\sum_{i \in \mathcal{F}_m} a_i d_i}{\sum_{i \in \mathcal{F}_m} a_i^2}.$$

Multiplying by the denominator yields

$$\left(\sum_{i \in \mathcal{F}_m} a_i^2 \right) \sum_{j \in \mathcal{U}_m^C} a_j d_j \geqslant \left(\sum_{j \in \mathcal{U}_m^C} a_j^2 \right) \sum_{i \in \mathcal{F}_m} a_i d_i. \tag{32}$$

In the case $\sigma_m \geqslant \tau_m$, we add the expression

$$\left(\sum_{j \in \mathcal{F}_{m+1}} a_j^2 \right) \sum_{i \in \mathcal{F}_{m+1}} a_i d_i - \left(\sum_{i \in \mathcal{U}_m^C} a_i^2 \right) \sum_{j \in \mathcal{U}_m^C} a_j d_j$$

to both sides of (32); after observing that the indices $\mathcal{U}_m^C$ added to $\mathcal{U}_m$ to obtain $\mathcal{U}_{m+1}$ are the same indices removed from $\mathcal{F}_m$ to obtain $\mathcal{F}_{m+1}$, we obtain

$$\left(\sum_{i \in \mathcal{F}_{m+1}} a_i^2 \right) \sum_{j \in \mathcal{F}_m} a_j d_j \geqslant \left(\sum_{j \in \mathcal{F}_m} a_j^2 \right) \sum_{i \in \mathcal{F}_{m+1}} a_i d_i.$$

This implies that $\bar{d}_m \geqslant \bar{d}_{m+1}$, which give (P1). The case (P2) can be analyzed in a similar fashion.

Next, we prove the following generalization:

(P3) If $\sigma_m \geqslant \tau_m$ and $\sigma_k < \tau_k$ for all $k \in (m, L)$, $L \leqslant M_1$, then $\bar{d}_m \geqslant \bar{d}_L$.
(P4) If $\sigma_m < \tau_m$ and $\sigma_k \geqslant \tau_k$ for all $k \in (m, L)$, $L \leqslant M_1$, then $\bar{d}_m \leqslant \bar{d}_L$.

Focusing first on (P3), it follows from (P1) that

$$\bar{d}_m \geqslant \bar{d}_{m+1} \quad \text{and} \quad \bar{d}_k \leqslant \bar{d}_{k+1} \text{ for } k \in (m, L). \tag{33}$$

We now show that $\bar{d}_m \geqslant \bar{d}_L$ (see Figure 1).

The proof is by induction. For $l = m+1$ we have $\bar{d}_m \geqslant \bar{d}_k$ for all $k \in [m, l]$. Suppose that for some $l \in (m, L)$, we have $\bar{d}_m \geqslant \bar{d}_k$ for all $k \in [m, l]$. We prove that $\bar{d}_m \geqslant \bar{d}_{l+1} \geqslant \bar{d}_l$. The last inequality $\bar{d}_{l+1} \geqslant \bar{d}_l$ follows directly from (33). To prove the first inequality $\bar{d}_m \geqslant \bar{d}_{l+1}$, we recall the definition (28) of $\bar{d}_m$, which can be rearranged as follows:

$$\sum_{i \in \mathcal{F}_m} (a_i d_i - a_i^2 \bar{d}_m) = 0. \tag{34}$$

We partition the terms in this sum into three separate terms. Observe that both $\mathcal{U}_m^C$ and $\mathcal{L}_m^C$ are (disjoint) subsets of $\mathcal{F}_m$. The remaining indices in $\mathcal{F}_m$ are denoted $\mathcal{G}$:

$$\mathcal{G} = \mathcal{F}_m \setminus [\mathcal{U}_m^C \cup \mathcal{L}_m^C].$$

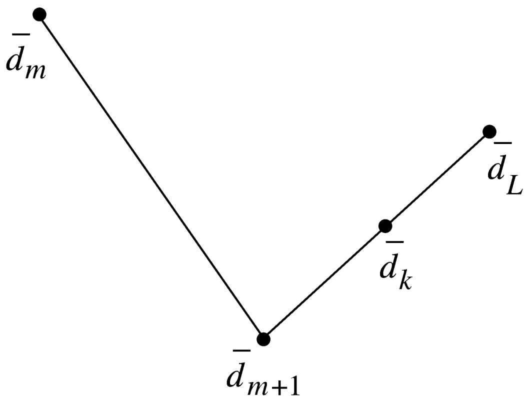

Figure 1. $\sigma_m \geqslant \tau_m$ and $\sigma_k < \tau_k$ for $m < k < L$.

Since the terms in sum (34) corresponding to $\mathcal{U}_m^C$ are used to form σ_m and the terms corresponding to $\mathcal{L}_m^C$ are used to form τ_m, (34) is equivalent to

$$\sigma_m - \tau_m + \sum_{i\in\mathcal{G}} (a_i d_i - a_i^2 \bar{d}_m) = 0.$$

By assumption, $\sigma_m \geqslant \tau_m$. Hence, $\sigma_m - \tau_m \geqslant 0$, which implies that

$$\sum_{i\in\mathcal{G}} (a_i d_i - a_i^2 \bar{d}_m) \leqslant 0. \tag{35}$$

By the induction hypothesis, $\bar{d}_m \geqslant \bar{d}_k$ for all $k \in [m,l]$. Since the set $\mathcal{L}_k^C$ is composed of indices for which $d_i \leqslant a_i \bar{d}_k$, the relation $\bar{d}_k \leqslant \bar{d}_m$ implies that $\mathcal{L}_k^C \subset \mathcal{L}_m^C$ for each $k \in [m,l]$. Since $\sigma_k < \tau_k$ for $k \in [m,l]$, it follows from (28) and (30) that

$$\mathcal{F}_k \backslash \mathcal{F}_{k+1} = \mathcal{L}_k^C \subset \mathcal{L}_m^C.$$

To summarize, for $k = m$, the set $\mathcal{F}_{m+1}$ is obtained from $\mathcal{F}_m$ by removing all the indices in $\mathcal{U}_m^C$, while for $k \in (m,l]$, $\mathcal{F}_{k+1}$ is obtained from $\mathcal{F}_k$ by removing indices that are elements of $\mathcal{L}_m^C$. Since $\mathcal{G}$ is the subset of $\mathcal{F}_m$ obtained by deleting the indices from $\mathcal{U}_m^C \cup \mathcal{L}_m^C$, we conclude that

$$\mathcal{G} \subset \mathcal{F}_{l+1} \subset \mathcal{G} \cup \mathcal{L}_m^C.$$

Hence for each $i \in \mathcal{F}_{l+1} \backslash \mathcal{G}$, we have $i \in \mathcal{L}_m^C$ and $d_i \leqslant a_i \bar{d}_m$, combining this with (35) gives

$$\sum_{i\in\mathcal{F}_{l+1}} (a_i d_i - a_i^2 \bar{d}_m) \leqslant \sum_{i\in\mathcal{G}} (a_i d_i - a_i^2 \bar{d}_m) \leqslant 0.$$

After dividing by $\sum_{i\in\mathcal{F}_{l+1}} a_i^2$, we obtain

$$\bar{d}_{l+1} = \frac{\sum_{i\in\mathcal{F}_{l+1}} a_i d_i}{\sum_{\mathcal{F}_{l+1}} a_i^2} \leqslant \bar{d}_m.$$

This completes the induction step. The (P4) case is similar, except that (P2) is used in place of (P1).

We now prove the lemma by contradiction. The first claim of the lemma is that $\bar{d}_m \geqslant \bar{d}_k$ for all $k \in (m, M_1]$ when $\sigma_m \geqslant \tau_m$. Let $L > m$ be the first index for which $\bar{d}_m < \bar{d}_L$. Let $l < L$ be the index closest to L with the property that $\sigma_l \geqslant \tau_l$. Since $\sigma_m \geqslant \tau_m$, such an index $l \geqslant m$ exists. Since $\sigma_l \geqslant \tau_l$ while $\sigma_k < \tau_k$ for all $k \in (l, L)$, it follows from (P3) that $\bar{d}_l \geqslant \bar{d}_L > \bar{d}_m$ This contradicts the fact that L was the smallest index with the property that $\bar{d}_L > \bar{d}_m$. The proof that $\bar{d}_m \leqslant \bar{d}_k$ for $k \in (m, M_1]$ when $\sigma_m < \tau_m$ is similar, using (P4) in place of (P3). □

LEMMA 4. *If $\Lambda(\beta)$ is multivalued, or $\Lambda(\beta)$ is single-valued and $\mathcal{F}_M$ is empty, then there exists $\epsilon > 0$ such that $\boldsymbol{x}(\alpha) = \boldsymbol{x}(\beta)$ for all $\alpha \in [\beta, \beta + \epsilon]$.*

Proof. If $\Lambda(\beta)$ is multivalued, then by Remark 1, $\boldsymbol{x}(\alpha) = \boldsymbol{x}(\beta)$ for all $\alpha > \beta, \alpha$ near β. If $\Lambda(\beta)$ is single-valued and $\mathcal{F}_M$ is empty, then for $\alpha > \beta$, α near β, define

$$\begin{aligned} \lambda(\alpha) &= \lambda(\beta) + (\alpha - \beta)\bar{d}_{M-1}, \\ \mathbf{x}(\alpha) &= \mathbf{x}(\beta). \end{aligned}$$

We show that for this choice of λ, the optimality conditions (16) are satisfied for $\alpha > \beta$, α near β.

If $i \in \mathcal{U}_M$ and $i \in \mathcal{U}_0$, then the relation

$$x_i(\beta) - y_i(\beta) + a_i \lambda(\beta) < 0$$

implies that the first inequality in (16) is satisfied for α near β. If $i \in \mathcal{U}_{m+1} \backslash \mathcal{U}_m$ for some $m \geqslant 0$, we have

$$x_i(\beta) - y_i(\beta) + a_i \lambda(\beta) = 0, \tag{36}$$

$\sigma_m \geqslant \tau_m$, and $d_i \geqslant a_i \bar{d}_m$. Since $\bar{d}_m \geqslant \bar{d}_k$ for all $k \in (m, M)$ by Lemma 3, $d_i \geqslant a_i \bar{d}_{M-1}$, or equivalently,

$$a_i \bar{d}_{M-1} - d_i \leqslant 0.$$

For the proposed solution, it follows that

$$x_i'(\alpha) - y_i'(\alpha) + a_i \lambda'(\alpha) = a_i \bar{d}_{M-1} - d_i \leqslant 0$$

Combining this with (36), we conclude that the first inequality in (16) is satisfied for all $\alpha > \beta$. The second inequality in (16) is analyzed in the same way. Since the first-order optimality conditions are necessary and sufficient for optimality, we conclude that the proposed $\mathbf{x}(\alpha)$ is the solution of (15) for $\alpha > \beta$, α near β. □

We now prove Theorem 2. The proof is constructive in the sense that we exhibit the solution. In particular, for $\alpha > \beta$ and α near β, we show that

$$\begin{aligned} x_i(\alpha) &= 1 \quad \text{for } i \in \mathcal{U}_M, \\ x_i(\alpha) &= 0 \quad \text{for } i \in \mathcal{L}_M, \\ x_i(\alpha) &= x_i(\beta) + (\alpha - \beta)(d_i - a_i \bar{d}_M) \quad \text{for } i \in \mathcal{F}_M, \end{aligned} \tag{37}$$

We simply check that this proposed solution satisfies the optimality conditions. In the case where $\mathcal{F}_M$ is empty, the optimality of the proposed solution follows from Lemma 4. Suppose that $\mathcal{F}_M$ is nonempty. If for some $i \in \mathcal{F}_M$, we have $x_i(\beta) = 1$, then since $i \notin \mathcal{U}_M$ and $\mathcal{U}_M^C$ is empty, it follows that $d_i < a_i \bar{d}_M$. By (37) $x_i(\alpha) < 1$ for $\alpha > \beta$. In the same fashion, if $i \in \mathcal{F}_M$ and $x_i(\beta) = 0$, then $x_i(\alpha) > 0$ for $\alpha > \beta$. Hence, for the proposed solution $0 < x_i(\alpha) < 1$ for all $i \in \mathcal{F}_M$ and $\alpha > \beta$. Also, for the proposed solution (37), $x_i'(\alpha) = d_i - a_i \bar{d}_M$, which matches the slope of the optimal solution given in (22) and (23). As a result, the equality in (16) holds.

Now consider the inequalities in (16). If $i \in \mathcal{U}_M$, then either $i \in \mathcal{U}_0$ and the first inequality in (16) is strict, or $i \in \mathcal{U}_{m+1} \backslash \mathcal{U}_m$ for some $m \geqslant 0$. In this latter case,

$$x_i(\beta) - y_i(\beta) + a_i \lambda(\beta) = 0, \tag{38}$$

$\sigma_m \geqslant \tau_m$, and $d_i \geqslant a_i \bar{d}_m$. Since $\bar{d}_m \geqslant \bar{d}_k$ for all $k \in (m, M]$ by Lemma 3, $d_i \geqslant a_i d_M$, or equivalently,

$$a_i \bar{d}_M - d_i \leqslant 0. \tag{39}$$

For the proposed solution, it follows from (22) that $\lambda'(\alpha) = \bar{d}_M$ for $\alpha > \beta$. Combining this with (39), we conclude that the derivative of the left side of (16) is $\leqslant 0$. Due to (38), the first inequality in (16) is satisfied for $\alpha > \beta$. The second inequality in (16) is treated in a similar manner. This completes the proof of Theorem 2.

5. Conclusions

To summarize, the projection (15) is evaluated by the following procedure:

5.1. PROJECTION ALGORITHM

1. Initialize $k = 0$, $\alpha_0 = 0$, $\mathbf{x}(0) = \mathbf{y}_0$. If $\Lambda(0)$ is single-valued, then proceed to step 2. Otherwise, α_1 is the largest value of $\beta > 0$ with the property that $\lambda_l(\alpha) \leqslant \lambda_u(\alpha)$ for all $\alpha \in [0, \beta]$, set $\mathbf{x}(\alpha) = \mathbf{x}(0)$ for $\alpha \in [0, \beta]$, increment k, and set $\lambda(\beta) = \lambda_l(\beta)$.
2. Initialize $\mathcal{U}_0$ and $\mathcal{L}_0$ using (26) and (27) with $\beta = \alpha_k$.

3. Perform the iteration (29) and (30) until both sets $\mathcal{U}_m^C$ and $\mathcal{L}_m^C$ are empty. Let $\mathcal{U}_M$ and $\mathcal{L}_M$ be the final sets. If $\mathcal{F}_M$ is nonempty, proceed to step 4. Otherwise, α_{k+1} is the largest value of $\beta \geqslant \alpha_k$ with the property that $\lambda_l(\alpha) \leqslant \lambda_u(\alpha)$ for all $\alpha \in [\alpha_k, \beta]$. Set $\mathbf{x}(\alpha) = \mathbf{x}(\alpha_k)$ for $\alpha \in [\alpha_k, \beta]$, set $\lambda(\beta) = \lambda_l(\beta)$, increment k, and branch to step 2.
4. Evaluate the slopes

$$\lambda'(\alpha) = \frac{\sum_{i \in \mathcal{F}_M} a_i d_i}{\sum_{i \in \mathcal{F}_M} a_i^2},$$

$$x_i'(\alpha) = d_i - a_i \lambda'(\alpha) \quad \text{for each } i \in \mathcal{F}_M,$$

$$x_i'(\alpha) = 0 \text{ otherwise.}$$

5. Using these slopes, evaluate $\mathbf{x}(\alpha)$ and $\lambda(\alpha)$ for $\alpha \geqslant \alpha_k$. Compute the next break point β that satisfies

$$x_i(\beta) = 0 \text{ or } 1 \text{ for some } i \in \mathcal{F}_M, \quad \text{or}$$

$$x_i(\beta) = y_i(\beta) - a_i \lambda(\beta), \quad i \in \mathcal{F}_M^c \quad \text{with } d_i \neq a_i \lambda'(\alpha).$$

6. Set $\alpha_{k+1} = \beta$, increment k, and branch to step 2.

This process is continued until we reach the last break point. For the graph partitioning problem (5), where the cost function is quadratic, the minimization between each break point amounts to minimizing a quadratic function of one variable. Moreover, due to the special form of the linear constraint in (5), it can be shown (see Appendix) that if $\mathcal{F}_M$ is empty at a break point β, then $\mathbf{x}(\alpha) = \mathbf{x}(\beta)$ for all $\alpha > \beta$. Hence, in this special case, the statement of the projection algorithm can be simplified further.

6. Appendix: Extreme Points for the Graph Partitioning Problem

Let us consider (5) where the linear constraint has the special form $\mathbf{1}^\top \mathbf{x} = m$, where $\mathbf{1}$ is the vector of ones and m is an integer. In this case, the extreme points of the constraint set are the vectors in $\mathbb{R}^n$ whose components contain m ones and $n - m$ zeros.

LEMMA A1. *If* $\mathbf{x}$ *is an extreme point for the feasible set* $\mathcal{K}$ *of* (5), *then for all* $\mathbf{a}, \mathbf{b} \in \mathcal{K}$, *we have* $(\mathbf{a} - \mathbf{x})^\top (\mathbf{b} - \mathbf{x}) \geqslant 0$.

Proof. First, suppose that $\mathbf{a}$ and $\mathbf{b}$ are extreme points of $\mathcal{K}$. Since $\mathbf{x}$ is an extreme point, $a_i < x_i$ only when $x_i = 1$, and in this case $b_i - x_i \leqslant 0$ since $b_i \leqslant 1$. Hence, the product $(a_i - x_i)(b_i - x_i)$ is nonnegative for each i. It follows that $(\mathbf{a} - \mathbf{x})^\top (\mathbf{b} - \mathbf{x}) \geqslant 0$. By the Krein–Milman theorem [3,p.181], $\mathcal{K}$ is a convex hull of its extreme points. Let χ_i, $i = 1, \ldots, N$, denote the extreme points of $\mathcal{K}$. Given arbitrary $\mathbf{a}$ and $\mathbf{b}\in\mathcal{K}$, there exist nonnegative scalars θ_i and μ_i, $i=1,\ldots,N$, such that

$$\sum_{i=1}^{N}\theta_i\chi_i=\mathbf{a} \quad\text{and}\quad \sum_{i=1}^{N}\mu_i\chi_i=\mathbf{b}, \quad\text{where } \sum_{i=1}^{N}\theta_i=1=\sum_{i=1}^{N}\mu_i.$$

Hence , we what

$$(\mathbf{a}-\mathbf{x})^\top(\mathbf{b}-\mathbf{x})=\left(\sum_{i=1}^{N}\theta_i(\chi_i-\mathbf{x})\right)^\top\sum_{j=1}^{N}\mu_j(\chi_j-\mathbf{x}).$$

Since each of the products $(\chi_i-\mathbf{x})^\top(\chi_j-\mathbf{x})$ is nonnegative, the proof is complete. □

The normal cone to $\mathcal{K}$ at the point $\mathbf{x}\in\mathcal{K}$ is defined by

$$N_{\mathcal{K}}(\mathbf{x})=\{\mathbf{z}\in\mathbb{R}^n\colon\ \mathbf{z}^\top(\mathbf{y}-\mathbf{x})\leqslant 0 \text{ for all } \mathbf{y}\in\mathcal{K}\}.$$

By Lemma, A1, it follows that $\mathcal{K}-\mathbf{x}\subset -N_{\mathcal{K}}(\mathbf{x})$ when $\mathbf{x}$ is an extreme point of $\mathcal{K}$. Also notice that the first-order optimality condition (6) describing the projection $P(\mathbf{y})$ is equivalent to the inclusion $\mathbf{y}-P(\mathbf{y})\in N_{\mathcal{K}}(P(\mathbf{y}))$.

LEMMA A2. *If $\mathbf{x}$ is an extreme point for the feasible set $\mathcal{K}$ of* (5), *and $P(\mathbf{y}_0+\beta\mathbf{d})=\mathbf{x}$, where $\mathbf{y}_0\in\mathcal{K}$ and $\beta>0$, then $P(\mathbf{y}_0+\alpha\mathbf{d})=\mathbf{x}$ for all $\alpha\geqslant\beta$.*

Proof. Since $\mathcal{K}-\mathbf{x}\subset -N_{\mathcal{K}}(\mathbf{x})$ and $\mathbf{y}_0\in\mathcal{K}$, it follows that

$$-(\mathbf{y}_0-\mathbf{x})\in N_{\mathcal{K}}(\mathbf{x}). \tag{40}$$

The assumption $P(\mathbf{y}_0+\beta\mathbf{d})=\mathbf{x}$ along with (6) implies that

$$\mathbf{y}_0+\beta\mathbf{d}-\mathbf{x}\in N_{\mathcal{K}}(\mathbf{x}). \tag{41}$$

The convexity of the normal cone combined with (40) and (41) give

$$\frac{1}{2}(-(\mathbf{y}_0-\mathbf{x}))+\frac{1}{2}(\mathbf{y}_0+\beta\mathbf{d}-\mathbf{x})=\frac{1}{2}\beta\mathbf{d}\in N_{\mathcal{K}}(\mathbf{x}).$$

Hence, all positive multiples of $\mathbf{d}$ lie in $N_{\mathcal{K}}(\mathbf{x})$. Since $N_{\mathcal{K}}(\mathbf{x})$ is a convex cone containing both $\mathbf{y}_0+\beta\mathbf{d}-\mathbf{x}$ and all positive multiples of $\mathbf{d}$,

$$\mathbf{y}_0+\alpha\mathbf{d}-\mathbf{x}\in N_{\mathcal{K}}(\mathbf{x})$$

for all $\alpha\geqslant\beta$. Consequently, by (6) $P(\mathbf{y}_0+\alpha\mathbf{d})=\boldsymbol{x}$ for all $\alpha\geqslant\beta$. □

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